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RESEARCH ARTICLE

On A Problem Associated with q- Starlike Functions of Order Υ

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ABSTRACT

In this particular paper , we prove a new result of Fekete Szegö inequality (an equality that gives the relation between first two coefficients of analytic functions) in the open unit disk E , by using a q – starlike function $S^*_{q,n}(\Upsilon)$[1]. For solving our problem , we use the concepts of quantum number $|j|_q$, quantum derivative $D_q\, f(z)$ and a q- differential operator $M^n_q\, f(z)$ that are defined by Jackson [9,10] , Abdullah Alsoboh and Maslina Darus [1]

Keywords- Analytic functions ,q – differential operator , quantum number, quantum derivative , q – starlike function , Fekete – Szegö Inequality , bounded analytic functions , concept of subordination
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INTRODUCTION

We are dealing with geometric function theory , which is a branch of complex analysis (a branch of mathematics which deals with those particular functions that are defined in some region of the complex plane). Geometric function theory deals with univalent functions , multivalent functions and analytic functions geometrically. Till now many mathematicians have worked on it. This was the result of their researches that in 19th century , a theorem called Riemann Mapping Theorem was proved by Koebe [12] , which was the base of geometric function theory . From this theorem , a conjecture arises that was given by Bieberbach [4] in 1916 and proved by Louis De Branges [5] in 1985 , which says that if $f \in S$, then the necessary condition for bounds of coefficients of f is $|a_n| \leq n \; \forall \; n \geq 2$. After the proof of this conjecture , it became "Bieberbach Conjecture de Branges theorem". When the mathematicians were tackling with this conjecture , then an inequality arises in 1933 that is called Fekete Szegö inequality [7].

To prove our result , firstly we will discuss some classes and some basic concepts which are as follows :-

Class of analytic functions:-

Class of analytic functions:-

This class is denoted by A , having functions of the type $f(z) = z + \sum_{j=2}^{\infty} a_j z^j$ and with the normalization conditions $f(0) = 0$, $f'(0) = 1$.

Class of univalent functions :-

This class is denoted by S , having functions of the type $f(z) = z + \sum_{j=2}^{\infty} a_j z^j$ and with the normalization conditions $f(0) = 0$, $f'(0) = 1$.

Class of bounded analytic functions or schwarzian functions :-

This class is defined by $w(z) = \sum_{n=1}^{\infty} c_n z^n$,following the conditions $w(0) = 0$ and $|w(z)| < 1$. For this function , Miller et. al. [14] gave the necessary and sufficient conditions that are

$$|c_1| \leq 1, \; |c_2| \leq 1-|c_1|^2$$

Class of Starlike Functions :-

This class is denoted by S^* . A function $f \in A$ is said to be a Starlike function if it is in starlike domain with respect to the origin. Its necessary and sufficient condition given by Duren [6] is

$$Re\left(\frac{zf'(z)}{f(z)}\right) > 0; z \in E; E = \{z \in C : |z| < 1\}.$$

Concept of subordination :-

This is our proposed approach , which is given by Lindelof [13]. In order to understand this concept , let us assume that we have two analytic functions $\{g(z)$ and $G(z)\}$ and one bounded analytic function $f(z)$ in such a way that

$$|f(z)| < 1 , f(0) = 0 \text{ and } g(z) = G\{f(z)\}; z \in E$$

then $g(z)$ is said to be subordinate to $G(z)$ and symbol for subordination is $\prec$.

Concept of quantum number and quantum derivative :-

quantum number, $|j|_q = \frac{q^{j-1}}{q-1}$; $0 < q < 1$, $z \neq 0$, $j \in N$

quantum derivative, $D_q f(z) = \frac{f(qz)-f(z)}{(q-1)z}$; $q \neq 0$, $z \neq 0$.

given by Jackson [9,10].

Concept of $q-$starlike functions of order Y and a $q-$differential operator :-

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$S^*_{q,n}(\Upsilon) = \Big\{f \in A : \text{Re}\left(\frac{z\, D_q\{M_q^n f(z)\}}{M_q^n f(z)}\right) > \Upsilon \;; z \in E\Big\}$; $\Upsilon \in [0,1)$, $q \in (0,1)$ and n ∈ N.

$M_q^0 f(z) = f(z)$, $M_q^1 f(z) = zD_q f(z) = z + \sum_{j=2}^{\infty} |j|_q\, a_j\, z^j$

$M_q^n f(z) = z\, D_q\{M_q^{n-1} f(z)\} = z + \sum_{j=2}^{\infty} |j|_q^{\,n}\, a_j\, z^j$

given by Abdullah Alsoboh and Maslina Darus [1].

By using all the above defined terms , we prove this inequality for the class $TA_{q,n}[\alpha,\beta,\Upsilon]$ which is defined as below

$$[1-(\alpha+\beta)]\,\frac{f(z)}{z} + \beta\,\frac{M_q^n f(z)}{z} + \alpha\left(\frac{z\, D_q\{M_q^n f(z)\}}{M_q^n f(z)}\right) \prec \phi(z).$$

As $\left(\frac{z\, D_q\{M_q^n f(z)\}}{M_q^n f(z)}\right) \prec \phi(z)$, given by Abdullah Alsoboh and Maslina Darus [1].

MAIN RESULTS

THEOREM-1:- Let $f(z) \in TA_{q,n}[\alpha,\beta,\Upsilon]$ and $\phi(z) = \frac{1+w(z)}{1-w(z)}$; $w(z)$ is a Schwarzian function , then

$$|\,a_3 - \mu a_2^2\,| \le \begin{cases} \frac{[2\{1-(\alpha+\beta)+(\beta-\alpha+\alpha|2|_q)|2|_q^n\}^2+4\alpha|2|_q^{2n}(|2|_q-1)]}{\{1-(\alpha+\beta)+(\beta-\alpha+\alpha|2|_q)|2|_q^n\}^2\{1-(\alpha+\beta)+(\beta-\alpha+\alpha|3|_q)|3|_q^n\}} - \frac{4\mu}{\{1-(\alpha+\beta)+(\beta-\alpha+\alpha|2|_q)|2|_q^n\}^2} ; \\ \qquad \mu \le \frac{\alpha|2|_q^{2n}(|2|_q-1)}{\{1-(\alpha+\beta)+(\beta-\alpha+\alpha|3|_q)|3|_q^n\}} \\ \frac{2}{\{1-(\alpha+\beta)+(\beta-\alpha+\alpha|3|_q)|3|_q^n\}} ; \\ \frac{\alpha|2|_q^{2n}(|2|_q-1)}{\{1-(\alpha+\beta)+(\beta-\alpha+\alpha|3|_q)|3|_q^n\}} \le \mu \le \frac{[\{1-(\alpha+\beta)+(\beta-\alpha+\alpha|2|_q)|2|_q^n\}^2+\alpha|2|_q^{2n}(|2|_q-1)]}{\{1-(\alpha+\beta)+(\beta-\alpha+\alpha|3|_q)|3|_q^n\}} ; \\ \frac{4\mu}{\{1-(\alpha+\beta)+(\beta-\alpha+\alpha|2|_q)|2|_q^n\}^2} - \frac{[2\{1-(\alpha+\beta)+(\beta-\alpha+\alpha|2|_q)|2|_q^n\}^2+4\alpha|2|_q^{2n}(|2|_q-1)]}{\{1-(\alpha+\beta)+(\beta-\alpha+\alpha|2|_q)|2|_q^n\}^2\{1-(\alpha+\beta)+(\beta-\alpha+\alpha|3|_q)|3|_q^n\}} ; \\ \qquad \mu \ge \frac{[\{1-(\alpha+\beta)+(\beta-\alpha+\alpha|2|_q)|2|_q^n\}^2+\alpha|2|_q^{2n}(|2|_q-1)]}{\{1-(\alpha+\beta)+(\beta-\alpha+\alpha|3|_q)|3|_q^n\}} . \end{cases}$$

The result is sharp.

PROOF :- By definition of $TA_{q,n}[\alpha,\beta,\Upsilon]$,

$$[1-(\alpha+\beta)]\,\frac{f(z)}{z} + \beta\,\frac{M_q^n f(z)}{z} + \alpha\left(\frac{z\, D_q\{M_q^n f(z)\}}{M_q^n f(z)}\right) = \frac{1+w(z)}{1-w(z)} \qquad \text{----- (1.1)}$$

By putting all the values in (1.1), we get

$$1 + \left[1-(\alpha+\beta)+(\beta-\alpha+\alpha|2|_q)|2|_q^n\right] a_2 z + \left[\{1-(\alpha+\beta)+(\beta-\alpha+\alpha|3|_q)|3|_q^n\}\, a_3 - \alpha|2|_q^{2n}(|2|_q-1)\; a_2^2\,\right] z^2 + \ldots$$
$$= 1 + 2\, c_1 z + 2\,(c_2 + c_1^2)\, z^2 + \ldots$$

By comparing the coefficients , we get

$$a_2 = \frac{2c_1}{[1-(\alpha+\beta)+(\beta-\alpha+\alpha|2|_q)|2|_q^n]} \quad \text{and}$$

$$a_3 = \frac{2c_2}{\{1-(\alpha+\beta)+(\beta-\alpha+\alpha|3|_q)|3|_q^n\}} + \frac{[2\{1-(\alpha+\beta)+(\beta-\alpha+\alpha|2|_q)|2|_q^n\}^2+4\alpha|2|_q^{2n}(|2|_q-1)]c_1^2}{\{1-(\alpha+\beta)+(\beta-\alpha+\alpha|2|_q)|2|_q^n\}^2\{1-(\alpha+\beta)+(\beta-\alpha+\alpha|3|_q)|3|_q^n\}} .$$

So, we get

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$$a_3 - \mu a_2^2 = \frac{2c_2}{\{1-(\alpha+\beta)+(\beta-\alpha+\alpha|3|_q)|3|_q^n\}} + \frac{\left[2\{1-(\alpha+\beta)+(\beta-\alpha+\alpha|2|_q)|2|_q^n\}^2+4\alpha|2|_q^{2n}(|2|_q-1)\right]c_1^2}{\{1-(\alpha+\beta)+(\beta-\alpha+\alpha|2|_q)|2|_q^n\}^2\{1-(\alpha+\beta)+(\beta-\alpha+\alpha|3|_q)|3|_q^n\}} - \mu\frac{4c_1^2}{\{1-(\alpha+\beta)+(\beta-\alpha+\alpha|2|_q)|2|_q^n\}^2}.$$

Applying mode on both sides and using $|c_2| \le 1-|c_1|^2$, we get

$$|a_3-\mu a_2^2| \le \frac{2}{\{1-(\alpha+\beta)+(\beta-\alpha+\alpha|3|_q)|3|_q^n\}} + \left\{\left|\frac{\left[2\{1-(\alpha+\beta)+(\beta-\alpha+\alpha|2|_q)|2|_q^n\}^2+4\alpha|2|_q^{2n}(|2|_q-1)\right]}{\{1-(\alpha+\beta)+(\beta-\alpha+\alpha|2|_q)|2|_q^n\}^2\{1-(\alpha+\beta)+(\beta-\alpha+\alpha|3|_q)|3|_q^n\}} - \mu\frac{4}{\{1-(\alpha+\beta)+(\beta-\alpha+\alpha|2|_q)|2|_q^n\}^2}\right| - \frac{2}{\{1-(\alpha+\beta)+(\beta-\alpha+\alpha|3|_q)|3|_q^n\}}\right\}|c_1|^2.$$

Case - 1 :- When $\mu \le \dfrac{\left[\{1-(\alpha+\beta)+(\beta-\alpha+\alpha|2|_q)|2|_q^n\}^2+2\alpha|2|_q^{2n}(|2|_q-1)\right]}{2\{1-(\alpha+\beta)+(\beta-\alpha+\alpha|3|_q)|3|_q^n\}}$.

Then,
$$|a_3-\mu a_2^2| \le \frac{2}{\{1-(\alpha+\beta)+(\beta-\alpha+\alpha|3|_q)|3|_q^n\}} + \left\{\frac{4\alpha|2|_q^{2n}(|2|_q-1)}{\{1-(\alpha+\beta)+(\beta-\alpha+\alpha|2|_q)|2|_q^n\}^2\{1-(\alpha+\beta)+(\beta-\alpha+\alpha|3|_q)|3|_q^n\}} - \frac{4\mu}{\{1-(\alpha+\beta)+(\beta-\alpha+\alpha|2|_q)|2|_q^n\}^2}\right\}|c_1|^2.$$

Subcase – 1 (a) :- If $\mu \le \dfrac{\alpha|2|_q^{2n}(|2|_q-1)}{\{1-(\alpha+\beta)+(\beta-\alpha+\alpha|3|_q)|3|_q^n\}}$

By using $|c_1| \le 1$, we get $|a_3-\mu a_2^2| \le$

$$\frac{\left[2\{1-(\alpha+\beta)+(\beta-\alpha+\alpha|2|_q)|2|_q^n\}^2+4\alpha|2|_q^{2n}(|2|_q-1)\right]}{\{1-(\alpha+\beta)+(\beta-\alpha+\alpha|2|_q)|2|_q^n\}^2\{1-(\alpha+\beta)+(\beta-\alpha+\alpha|3|_q)|3|_q^n\}} - \frac{4\mu}{\{1-(\alpha+\beta)+(\beta-\alpha+\alpha|2|_q)|2|_q^n\}^2}$$

----- (1.2)

Subcase – 1 (b) :- If $\mu \ge \dfrac{\alpha|2|_q^{2n}(|2|_q-1)}{\{1-(\alpha+\beta)+(\beta-\alpha+\alpha|3|_q)|3|_q^n\}}$

Then, $|a_3-\mu a_2^2| \le \dfrac{2}{\{1-(\alpha+\beta)+(\beta-\alpha+\alpha|3|_q)|3|_q^n\}}$ -------- (1.3)

Case – 2 :- When $\mu \ge \dfrac{\left[\{1-(\alpha+\beta)+(\beta-\alpha+\alpha|2|_q)|2|_q^n\}^2+2\alpha|2|_q^{2n}(|2|_q-1)\right]}{2\{1-(\alpha+\beta)+(\beta-\alpha+\alpha|3|_q)|3|_q^n\}}$

Then,
$$|a_3-\mu a_2^2| \le \frac{2}{\{1-(\alpha+\beta)+(\beta-\alpha+\alpha|3|_q)|3|_q^n\}} + \left\{\frac{4\mu}{\{1-(\alpha+\beta)+(\beta-\alpha+\alpha|2|_q)|2|_q^n\}^2} - \frac{4\left[\{1-(\alpha+\beta)+(\beta-\alpha+\alpha|2|_q)|2|_q^n\}^2+\alpha|2|_q^{2n}(|2|_q-1)\right]}{\{1-(\alpha+\beta)+(\beta-\alpha+\alpha|2|_q)|2|_q^n\}^2\{1-(\alpha+\beta)+(\beta-\alpha+\alpha|3|_q)|3|_q^n\}}\right\}|c_1|^2.$$

Subcase – 2 (a) :- If $\mu \ge \dfrac{\left[\{1-(\alpha+\beta)+(\beta-\alpha+\alpha|2|_q)|2|_q^n\}^2+\alpha|2|_q^{2n}(|2|_q-1)\right]}{\{1-(\alpha+\beta)+(\beta-\alpha+\alpha|3|_q)|3|_q^n\}}$

By using $|c_1| \le 1$, we get

$$|a_3-\mu a_2^2| \le \frac{4\mu}{\{1-(\alpha+\beta)+(\beta-\alpha+\alpha|2|_q)|2|_q^n\}^2} - \frac{2\left[\{1-(\alpha+\beta)+(\beta-\alpha+\alpha|2|_q)|2|_q^n\}^2+2\alpha|2|_q^{2n}(|2|_q-1)\right]}{\{1-(\alpha+\beta)+(\beta-\alpha+\alpha|2|_q)|2|_q^n\}^2\{1-(\alpha+\beta)+(\beta-\alpha+\alpha|3|_q)|3|_q^n\}}$$

-------- (1.4)

Subcase – 2 (b) :- If $\mu \le \dfrac{\left[\{1-(\alpha+\beta)+(\beta-\alpha+\alpha|2|_q)|2|_q^n\}^2+\alpha|2|_q^{2n}(|2|_q-1)\right]}{\{1-(\alpha+\beta)+(\beta-\alpha+\alpha|3|_q)|3|_q^n\}}$

Then, $|a_3-\mu a_2^2| \le \dfrac{2}{\{1-(\alpha+\beta)+(\beta-\alpha+\alpha|3|_q)|3|_q^n\}}$ -------- (1.5)

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Combining (1.2), (1.3), (1.4) and (1.5) we get the required result.

Extremal : For first and third equations, extremal is

$$f(z) = z\,[1 + az]^n$$

where $a = \dfrac{2\{1-(\alpha+\beta)+(\beta-\alpha+\alpha|3|_q)|3|_q^n\}-2\left[\{1-(\alpha+\beta)+(\beta-\alpha+\alpha|2|_q)|2|_q^n\}^2+2\alpha|2|_q^{2n}(|2|_q-1)\right]}{\{1-(\alpha+\beta)+(\beta-\alpha+\alpha|2|_q)|2|_q^n\}\{1-(\alpha+\beta)+(\beta-\alpha+\alpha|3|_q)|3|_q^n\}}$

and $n = \dfrac{\{1-(\alpha+\beta)+(\beta-\alpha+\alpha|3|_q)|3|_q^n\}}{\{1-(\alpha+\beta)+(\beta-\alpha+\alpha|3|_q)|3|_q^n\}-\left[\{1-(\alpha+\beta)+(\beta-\alpha+\alpha|2|_q)|2|_q^n\}^2+2\alpha|2|_q^{2n}(|2|_q-1)\right]}$.

For second equation, extremal is

$$f(z) = z\,[1 + 2z^2]^{\frac{1}{\{1-(\alpha+\beta)+(\beta-\alpha+\alpha|3|_q)|3|_q^n\}}} .$$

COROLLARY- 2:- $TA_{q,n}[1,0,\Upsilon] = S^*_{q,n}[\Upsilon]$; as by substituting α = 1 and $\beta = 0$, the result becomes

$$|a_3-\mu a_2^2| \le \begin{cases} \dfrac{2(|2|_q-1)+4}{|3|_q^n(|3|_q-1)(|2|_q-1)} - \dfrac{4\mu}{(|2|_q-1)^2|2|_q^{2n}} \ ; \ \mu \le \dfrac{(|2|_q-1)|2|_q^{2n}}{|3|_q^n(|3|_q-1)} \ ; \\ \dfrac{2}{|3|_q^n(|3|_q-1)} \ ; \ \dfrac{(|2|_q-1)|2|_q^{2n}}{|3|_q^n(|3|_q-1)} \le \mu \le \dfrac{(|2|_q-1)|2|_q^{2n+1}}{|3|_q^n(|3|_q-1)} \ ; \\ \dfrac{4\mu}{(|2|_q-1)^2|2|_q^{2n}} - \dfrac{2(|2|_q-1)+4}{|3|_q^n(|3|_q-1)(|2|_q-1)} \ ; \ \mu \ge \dfrac{(|2|_q-1)|2|_q^{2n+1}}{|3|_q^n(|3|_q-1)} . \end{cases}$$

which is the required result given by Abdullah Alsoboh and Maslina Darus [1].

COROLLARY- 3:- $TA_{q,n}[\alpha,\beta,\Upsilon] = TS^*_{q,n}[\alpha,\Upsilon]$; as by substituting $\alpha + \beta$ =1, the result becomes

$$|a_3-\mu a_2^2| \le$$

$$\begin{cases} \dfrac{2\{(1-2\alpha+\alpha|2|_q)^2+2\alpha(|2|_q-1)\}}{(1-2\alpha+\alpha|3|_q)(1-2\alpha+\alpha|2|_q)^2|3|_q^n} - \dfrac{4\mu}{(1-2\alpha+\alpha|2|_q)^2\,|2|_q^{2n}} \ ; \mu \le \dfrac{\alpha|2|_q^{2n}(|2|_q-1)}{(1-2\alpha+\alpha|3|_q)|3|_q^n} \ ; \\ \dfrac{2}{(1-2\alpha+\alpha|3|_q)|3|_q^n} \ ; \dfrac{\alpha|2|_q^{2n}(|2|_q-1)}{(1-2\alpha+\alpha|3|_q)|3|_q^n} \le \mu \le \dfrac{|2|_q^{2n}\{(1-2\alpha+\alpha|2|_q)^2+\alpha(|2|_q-1)\}}{(1-2\alpha+\alpha|3|_q)|3|_q^n} \ ; \\ \dfrac{4\mu}{(1-2\alpha+\alpha|2|_q)^2\,|2|_q^{2n}} - \dfrac{2\{(1-2\alpha+\alpha|2|_q)^2+2\alpha(|2|_q-1)\}}{(1-2\alpha+\alpha|3|_q)(1-2\alpha+\alpha|2|_q)^2|3|_q^n} \ ; \mu \ge \dfrac{|2|_q^{2n}\{(1-2\alpha+\alpha|2|_q)^2+\alpha(|2|_q-1)\}}{(1-2\alpha+\alpha|3|_q)|3|_q^n} . \end{cases}$$

which is same as the result proved by Gurmeet Singh and Misha Rani [22].

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THEOREM-4:- Let $f(z) \in TA_{q,n}[\alpha,\beta,\Upsilon,\delta]$ and $\phi(z) = \left(\frac{1+w(z)}{1-w(z)}\right)^{\delta}$; $w(z)$ is a Schwarzian function , then

$$| a_3 - \mu a_2^2 | \leq \begin{cases} \frac{\left[2\delta^2\{1-(\alpha+\beta)+(\beta-\alpha+\alpha|2|_q)|2|_q^n\}^2+4\alpha|2|_q^{2n}(|2|_q-1)\right]}{\{1-(\alpha+\beta)+(\beta-\alpha+\alpha|2|_q)|2|_q^n\}^2\{1-(\alpha+\beta)+(\beta-\alpha+\alpha|3|_q)|3|_q^n\}} - \frac{4\mu\delta^2}{\{1-(\alpha+\beta)+(\beta-\alpha+\alpha|2|_q)|2|_q^n\}^2} ; \\ \mu \leq \frac{(\delta^2-\delta)\{1-(\alpha+\beta)+(\beta-\alpha+\alpha|2|_q)|2|_q^n\}^2+2\alpha|2|_q^{2n}(|2|_q-1)}{2\delta^2\{1-(\alpha+\beta)+(\beta-\alpha+\alpha|3|_q)|3|_q^n\}} \\ \frac{2\delta}{\{1-(\alpha+\beta)+(\beta-\alpha+\alpha|3|_q)|3|_q^n\}} ; \\ \frac{(\delta^2-\delta)\{1-(\alpha+\beta)+(\beta-\alpha+\alpha|2|_q)|2|_q^n\}^2+2\alpha|2|_q^{2n}(|2|_q-1)}{2\delta^2\{1-(\alpha+\beta)+(\beta-\alpha+\alpha|3|_q)|3|_q^n\}} \leq \mu \leq \frac{\left[(\delta^2+\delta)\{1-(\alpha+\beta)+(\beta-\alpha+\alpha|2|_q)|2|_q^n\}^2+2\alpha|2|_q^{2n}(|2|_q-1)\right]}{2\delta^2\{1-(\alpha+\beta)+(\beta-\alpha+\alpha|3|_q)|3|_q^n\}} ; \\ \frac{4\mu\delta^2}{\{1-(\alpha+\beta)+(\beta-\alpha+\alpha|2|_q)|2|_q^n\}^2} - \frac{\left[2\delta^2\{1-(\alpha+\beta)+(\beta-\alpha+\alpha|2|_q)|2|_q^n\}^2+4\alpha|2|_q^{2n}(|2|_q-1)\right]}{\{1-(\alpha+\beta)+(\beta-\alpha+\alpha|2|_q)|2|_q^n\}^2\{1-(\alpha+\beta)+(\beta-\alpha+\alpha|3|_q)|3|_q^n\}} ; \\ \mu \geq \frac{\left[(\delta^2+\delta)\{1-(\alpha+\beta)+(\beta-\alpha+\alpha|2|_q)|2|_q^n\}^2+2\alpha|2|_q^{2n}(|2|_q-1)\right]}{2\delta^2\{1-(\alpha+\beta)+(\beta-\alpha+\alpha|3|_q)|3|_q^n\}} . \end{cases}$$

The result is sharp.

PROOF :- By definition of $TA_{q,n}[\alpha,\beta,\Upsilon,\delta]$,

$$[1-(\alpha+\beta)]\frac{f(z)}{z} + \beta\frac{M_q^n f(z)}{z} + \alpha\left(\frac{z\,D_q\{M_q^n f(z)\}}{M_q^n f(z)}\right) = \left(\frac{1+w(z)}{1-w(z)}\right)^{\delta} \qquad \text{-------(4.1)}$$

By putting all the values in (3.1), we get

$$1 + \left[1-(\alpha+\beta)+(\beta-\alpha+\alpha|2|_q)|2|_q^n\right] a_2 z$$
$$+\left[\{1-(\alpha+\beta)+(\beta-\alpha+\alpha|3|_q)|3|_q^n\} a_3 - \alpha|2|_q^{2n}(|2|_q-1)\ a_2^2\right]z^2 + \ldots$$
$$= 1 + 2\delta\, c_1 z + 2\,(\delta c_2 + \delta^2 c_1^2)\, z^2 + \ldots$$

By comparing the coefficients , we get

$$a_2 = \frac{2\delta c_1}{\left[1-(\alpha+\beta)+(\beta-\alpha+\alpha|2|_q)|2|_q^n\right]} \quad \text{and}$$

$$a_3 = \frac{2\delta c_2}{\{1-(\alpha+\beta)+(\beta-\alpha+\alpha|3|_q)|3|_q^n\}} + \frac{\left[2\delta^2\{1-(\alpha+\beta)+(\beta-\alpha+\alpha|2|_q)|2|_q^n\}^2+4\alpha\delta^2|2|_q^{2n}(|2|_q-1)\right]c_1^2}{\{1-(\alpha+\beta)+(\beta-\alpha+\alpha|2|_q)|2|_q^n\}^2\{1-(\alpha+\beta)+(\beta-\alpha+\alpha|3|_q)|3|_q^n\}} .$$

So, we get

$$a_3 - \mu\, a_2^2 = \frac{2\delta c_2}{\{1-(\alpha+\beta)+(\beta-\alpha+\alpha|3|_q)|3|_q^n\}} + \frac{\left[2\delta^2\{1-(\alpha+\beta)+(\beta-\alpha+\alpha|2|_q)|2|_q^n\}^2+4\alpha\delta^2|2|_q^{2n}(|2|_q-1)\right]c_1^2}{\{1-(\alpha+\beta)+(\beta-\alpha+\alpha|2|_q)|2|_q^n\}^2\{1-(\alpha+\beta)+(\beta-\alpha+\alpha|3|_q)|3|_q^n\}}$$
$$-\mu\frac{4\delta^2 c_1^2}{\{1-(\alpha+\beta)+(\beta-\alpha+\alpha|2|_q)|2|_q^n\}^2} .$$

Applying mode on both sides and using $|c_2| \leq 1 - |c_1|^2$, we get

$$| a_3 - \mu a_2^2 | \leq \frac{2\delta}{\{1-(\alpha+\beta)+(\beta-\alpha+\alpha|3|_q)|3|_q^n\}}$$
$$+\left\{\left|\frac{\left[2\delta^2\{1-(\alpha+\beta)+(\beta-\alpha+\alpha|2|_q)|2|_q^n\}^2+4\alpha\delta^2|2|_q^{2n}(|2|_q-1)\right]}{\{1-(\alpha+\beta)+(\beta-\alpha+\alpha|2|_q)|2|_q^n\}^2\{1-(\alpha+\beta)+(\beta-\alpha+\alpha|3|_q)|3|_q^n\}} - \mu\frac{4\delta^2}{\{1-(\alpha+\beta)+(\beta-\alpha+\alpha|2|_q)|2|_q^n\}^2}\right| - \frac{2\delta}{\{1-(\alpha+\beta)+(\beta-\alpha+\alpha|3|_q)|3|_q^n\}}\right\}|c_1|^2.$$

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Case - 1 :- When $\mu \leq \frac{\left[\{1-(\alpha+\beta)+(\beta-\alpha+\alpha|2|_q)|2|_q^n\}^2+2\alpha|2|_q^{2n}(|2|_q-1)\right]}{2\{1-(\alpha+\beta)+(\beta-\alpha+\alpha|3|_q)|3|_q^n\}}$.

Then, $|a_3-\mu a_2^2| \leq \frac{2\delta}{\{1-(\alpha+\beta)+(\beta-\alpha+\alpha|3|_q)|3|_q^n\}}$

$$+\left\{\frac{2(\delta^2-\delta)\{1-(\alpha+\beta)+(\beta-\alpha+\alpha|2|_q)|2|_q^n\}^2+4\alpha|2|_q^{2n}(|2|_q-1)}{\{1-(\alpha+\beta)+(\beta-\alpha+\alpha|2|_q)|2|_q^n\}^2\{1-(\alpha+\beta)+(\beta-\alpha+\alpha|3|_q)|3|_q^n\}}-\frac{4\mu\delta^2}{\{1-(\alpha+\beta)+(\beta-\alpha+\alpha|2|_q)|2|_q^n\}^2}\right\}|c_1|^2 .$$

Subcase – 1 (a) :- If $\mu \leq \frac{(\delta^2-\delta)\{1-(\alpha+\beta)+(\beta-\alpha+\alpha|2|_q)|2|_q^n\}^2+2\alpha|2|_q^{2n}(|2|_q-1)}{2\delta^2\{1-(\alpha+\beta)+(\beta-\alpha+\alpha|3|_q)|3|_q^n\}}$

By using $|c_1| \leq 1$, we get $|a_3-\mu a_2^2| \leq$

$$\frac{\left[2\delta^2\{1-(\alpha+\beta)+(\beta-\alpha+\alpha|2|_q)|2|_q^n\}^2+4\alpha|2|_q^{2n}(|2|_q-1)\right]}{\{1-(\alpha+\beta)+(\beta-\alpha+\alpha|2|_q)|2|_q^n\}^2\{1-(\alpha+\beta)+(\beta-\alpha+\alpha|3|_q)|3|_q^n\}}-\frac{4\mu\delta^2}{\{1-(\alpha+\beta)+(\beta-\alpha+\alpha|2|_q)|2|_q^n\}^2}$$

----- (4.2)

Subcase – 1 (b) :- If $\mu \geq \frac{(\delta^2-\delta)\{1-(\alpha+\beta)+(\beta-\alpha+\alpha|2|_q)|2|_q^n\}^2+2\alpha|2|_q^{2n}(|2|_q-1)}{2\delta^2\{1-(\alpha+\beta)+(\beta-\alpha+\alpha|3|_q)|3|_q^n\}}$

Then, $|a_3-\mu a_2^2| \leq \frac{2\delta}{\{1-(\alpha+\beta)+(\beta-\alpha+\alpha|3|_q)|3|_q^n\}}$ -------- (4.3)

Case – 2 :- When $\mu \geq \frac{\left[\{1-(\alpha+\beta)+(\beta-\alpha+\alpha|2|_q)|2|_q^n\}^2+2\alpha|2|_q^{2n}(|2|_q-1)\right]}{2\{1-(\alpha+\beta)+(\beta-\alpha+\alpha|3|_q)|3|_q^n\}}$

Then, $|a_3-\mu a_2^2| \leq \frac{2\delta}{\{1-(\alpha+\beta)+(\beta-\alpha+\alpha|3|_q)|3|_q^n\}}$

$$+\left\{\frac{4\mu\delta^2}{\{1-(\alpha+\beta)+(\beta-\alpha+\alpha|2|_q)|2|_q^n\}^2}-\frac{2(\delta^2+\delta)\{1-(\alpha+\beta)+(\beta-\alpha+\alpha|2|_q)|2|_q^n\}^2+4\alpha|2|_q^{2n}(|2|_q-1)}{\{1-(\alpha+\beta)+(\beta-\alpha+\alpha|2|_q)|2|_q^n\}^2\{1-(\alpha+\beta)+(\beta-\alpha+\alpha|3|_q)|3|_q^n\}}\right\}|c_1|^2 .$$

Subcase – 2 (a) :- If $\mu \geq \frac{\left[(\delta^2+\delta)\{1-(\alpha+\beta)+(\beta-\alpha+\alpha|2|_q)|2|_q^n\}^2+2\alpha|2|_q^{2n}(|2|_q-1)\right]}{2\delta^2\{1-(\alpha+\beta)+(\beta-\alpha+\alpha|3|_q)|3|_q^n\}}$

By using $|c_1| \leq 1$, we get

$$|a_3-\mu a_2^2| \leq \frac{4\mu\delta^2}{\{1-(\alpha+\beta)+(\beta-\alpha+\alpha|2|_q)|2|_q^n\}^2}-\frac{2\left[\delta^2\{1-(\alpha+\beta)+(\beta-\alpha+\alpha|2|_q)|2|_q^n\}^2+2\alpha|2|_q^{2n}(|2|_q-1)\right]}{\{1-(\alpha+\beta)+(\beta-\alpha+\alpha|2|_q)|2|_q^n\}^2\{1-(\alpha+\beta)+(\beta-\alpha+\alpha|3|_q)|3|_q^n\}}$$

-------- (4.4)

Subcase – 2 (b) :- If $\mu \leq \frac{\left[(\delta^2+\delta)\{1-(\alpha+\beta)+(\beta-\alpha+\alpha|2|_q)|2|_q^n\}^2+2\alpha|2|_q^{2n}(|2|_q-1)\right]}{2\delta^2\{1-(\alpha+\beta)+(\beta-\alpha+\alpha|3|_q)|3|_q^n\}}$

Then, $|a_3-\mu a_2^2| \leq \frac{2\delta}{\{1-(\alpha+\beta)+(\beta-\alpha+\alpha|3|_q)|3|_q^n\}}$ -------- (4.5)

Combining (4.2), (4.3), (4.4) and (4.5) we get the required result.

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Extremal : For first and third equations, extremal is

$$f(z) = z\,[1 + az]^n$$

where $a = \frac{2\delta^2\{1-(\alpha+\beta)+(\beta-\alpha+\alpha|3|_q)|3|_q^n\}-2\left[\delta^2\{1-(\alpha+\beta)+(\beta-\alpha+\alpha|2|_q)|2|_q^n\}^2+2\alpha|2|_q^{2n}(|2|_q-1)\right]}{\delta\{1-(\alpha+\beta)+(\beta-\alpha+\alpha|2|_q)|2|_q^n\}\{1-(\alpha+\beta)+(\beta-\alpha+\alpha|3|_q)|3|_q^n\}}$

and $n = \frac{2\delta^2\{1-(\alpha+\beta)+(\beta-\alpha+\alpha|3|_q)|3|_q^n\}}{2\delta^2\{1-(\alpha+\beta)+(\beta-\alpha+\alpha|3|_q)|3|_q^n\}-2\left[\delta^2\{1-(\alpha+\beta)+(\beta-\alpha+\alpha|2|_q)|2|_q^n\}^2+2\alpha|2|_q^{2n}(|2|_q-1)\right]}$

.

For second equation, extremal is

$$f(z) = z\,[1 + 2\delta z^2]^{\frac{1}{\{1-(\alpha+\beta)+(\beta-\alpha+\alpha|3|_q)|3|_q^n\}}}\ .$$

COROLLARY-5:- $TA_{q,n}[\alpha,\beta,\Upsilon,\delta] = TA_{q,n}[\alpha,\beta,\Upsilon]$, as by putting $\delta = 1$, the result becomes

$$|a_3-\mu a_2^2| \le \begin{cases} \frac{\left[2\{1-(\alpha+\beta)+(\beta-\alpha+\alpha|2|_q)|2|_q^n\}^2+4\alpha|2|_q^{2n}(|2|_q-1)\right]}{\{1-(\alpha+\beta)+(\beta-\alpha+\alpha|2|_q)|2|_q^n\}^2\{1-(\alpha+\beta)+(\beta-\alpha+\alpha|3|_q)|3|_q^n\}} - \frac{4\mu}{\{1-(\alpha+\beta)+(\beta-\alpha+\alpha|2|_q)|2|_q^n\}^2}\ ; \\ \qquad \mu \le \frac{\alpha|2|_q^{2n}(|2|_q-1)}{\{1-(\alpha+\beta)+(\beta-\alpha+\alpha|3|_q)|3|_q^n\}} \\ \frac{2}{\{1-(\alpha+\beta)+(\beta-\alpha+\alpha|3|_q)|3|_q^n\}}\ ; \\ \frac{\alpha|2|_q^{2n}(|2|_q-1)}{\{1-(\alpha+\beta)+(\beta-\alpha+\alpha|3|_q)|3|_q^n\}} \le \mu \le \frac{\left[\{1-(\alpha+\beta)+(\beta-\alpha+\alpha|2|_q)|2|_q^n\}^2+\alpha|2|_q^{2n}(|2|_q-1)\right]}{\{1-(\alpha+\beta)+(\beta-\alpha+\alpha|3|_q)|3|_q^n\}}\ ; \\ \frac{4\mu}{\{1-(\alpha+\beta)+(\beta-\alpha+\alpha|2|_q)|2|_q^n\}^2} - \frac{\left[2\{1-(\alpha+\beta)+(\beta-\alpha+\alpha|2|_q)|2|_q^n\}^2+4\alpha|2|_q^{2n}(|2|_q-1)\right]}{\{1-(\alpha+\beta)+(\beta-\alpha+\alpha|2|_q)|2|_q^n\}^2\{1-(\alpha+\beta)+(\beta-\alpha+\alpha|3|_q)|3|_q^n\}}\ ; \\ \qquad \mu \ge \frac{\left[\{1-(\alpha+\beta)+(\beta-\alpha+\alpha|2|_q)|2|_q^n\}^2+\alpha|2|_q^{2n}(|2|_q-1)\right]}{\{1-(\alpha+\beta)+(\beta-\alpha+\alpha|3|_q)|3|_q^n\}}\ . \end{cases}$$

which is same as $TA_{q,n}[\alpha,\beta,\Upsilon]$.

COROLLARY-6:- T $A_{q,n}[1,0,\Upsilon,1] = S^*_{q,n}[\Upsilon]$, as by putting α =1, $\beta = 0$ and $\delta = 1$, the result becomes

$$|a_3-\mu a_2^2| \le \begin{cases} \frac{2(|2|_q-1)+4}{|3|_q^n(|3|_q-1)(|2|_q-1)} - \frac{4\mu}{(|2|_q-1)^2|2|_q^{2n}}\ ;\ \mu \le \frac{(|2|_q-1)|2|_q^{2n}}{|3|_q^n(|3|_q-1)}\ ; \\ \frac{2}{|3|_q^n(|3|_q-1)}\ ;\ \frac{(|2|_q-1)|2|_q^{2n}}{|3|_q^n(|3|_q-1)} \le \mu \le \frac{(|2|_q-1)|2|_q^{2n+1}}{|3|_q^n(|3|_q-1)}\ ; \\ \frac{4\mu}{(|2|_q-1)^2|2|_q^{2n}} - \frac{2(|2|_q-1)+4}{|3|_q^n(|3|_q-1)(|2|_q-1)}\ ;\ \mu \ge \frac{(|2|_q-1)|2|_q^{2n+1}}{|3|_q^n(|3|_q-1)}\ . \end{cases}$$

which is same as that of $S^*_{q,n}\ [\Upsilon]$ given by Abdullah Alsoboh and Maslina Darus [1].

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COROLLARY-7:- $T\,A_{q,n}[\alpha,\beta,\Upsilon,\delta] = TS^{*}_{q,n}[\alpha,\Upsilon]$, as by putting $\alpha+\beta=1$ and $\delta=1$, the result becomes

$$|a_3-\mu a_2^2| \le \begin{cases} \dfrac{2\{(1-2\alpha+\alpha\,|2|_q)^2+2\alpha(|2|_q-1)\}}{(1-2\alpha+\alpha|3|_q)(1-2\alpha+\alpha\,|2|_q)^2|3|_q^n} - \dfrac{4\mu}{(1-2\alpha+\alpha\,|2|_q)^2\,|2|_q^{2n}} ; \mu \le \dfrac{\alpha|2|_q^{2n}(|2|_q-1)}{(1-2\alpha+\alpha|3|_q)|3|_q^n} ; \\ \dfrac{2}{(1-2\alpha+\alpha|3|_q)|3|_q^n} ; \dfrac{\alpha|2|_q^{2n}(|2|_q-1)}{(1-2\alpha+\alpha|3|_q)|3|_q^n} \le \mu \le \dfrac{|2|_q^{2n}\{(1-2\alpha+\alpha\,|2|_q)^2+\alpha(|2|_q-1)\}}{(1-2\alpha+\alpha|3|_q)|3|_q^n} ; \\ \dfrac{4\mu}{(1-2\alpha+\alpha\,|2|_q)^2\,|2|_q^{2n}} - \dfrac{2\{(1-2\alpha+\alpha\,|2|_q)^2+2\alpha(|2|_q-1)\}}{(1-2\alpha+\alpha|3|_q)(1-2\alpha+\alpha\,|2|_q)^2|3|_q^n} ; \mu \ge \dfrac{|2|_q^{2n}\{(1-2\alpha+\alpha\,|2|_q)^2+\alpha(|2|_q-1)\}}{(1-2\alpha+\alpha|3|_q)|3|_q^n} . \end{cases}$$

which is same as the result proved by Gurmeet Singh and Misha Rani [22].

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