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RESEARCH ARTICLE

A Generalized Sub Class of Univalent Starlike Functions with a Linear Operator

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ABSTRACT

This paper deals with a new class T* (α,β,λ) which is a subclass of uniformly starlike functions involving a linear operator L(a,b;c). Coefficients inequality, Distortion theorem, Extreme points, Radius of starlikeness and radius of convexity for functions belonging to this class are also obtained.

Keywords- Univalent, starlike, convex, analytic, linear operator.

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INTRODUCTION

Let S denote the class of functions of the form

$$(1.1)\qquad f(z) = z + \sum_{n=2}^{\infty} a_n z^n,$$

Which are analytic and univalentin the unit open disk Δ= {z: |z|<1}.

Let T be the subclass of S consisting of functions of the form

$$(1.2)\qquad f(z) = z - \sum_{n=2}^{\infty} a_n z^n, a_n \geq 0, \forall n \geq 2.$$

Which was introduced and studied by Silvarman (1975).

Now, we consider a function $\phi(a,b;c;z)$ as

$$(1.3)\qquad \phi(a,b;c;z) = z + \sum_{n=2}^{\infty} \frac{(a)_{n-1}(b)_{n-1}}{(n-1)!(c)_{n-1}} z^{n+k}\ for c \neq 0,-1,\ldots., a,b \neq -1, z \in \Delta.$$

Where $(\delta)_n$ is the Pochhammer symbol defined by

$$(1.4)\ (\delta)_n = \frac{\Gamma(n+\delta)}{\Gamma(\delta)} = \begin{cases} 1, n=0 \\ \delta(\delta+1)\ldots(\delta+n-1), n \in N. \end{cases}$$

Carlson and Shaffer (2002) introduced a linear operator L(a;c) which is defined as

$$L(a;c) f(z) = \phi(a;c;z) * f(z)$$

$$= z + \sum_{n=2}^{\infty} \frac{(a)_{n-1}}{(c)_{n-1}} a_n z^n, z \in \Delta$$

Where * stands for the Hadamard product of two power series

$$\phi(z) = \sum_{n=2}^{\infty} \phi_n z^n \text{ and } \varphi(z) = \sum_{n=2}^{\infty} \varphi_n z^n$$

defined by

$$(\phi * \varphi) z = \sum_{n=2}^{\infty} \phi_n \varphi_n z^n$$

Now we consider a linear operator L(a,b;c) as

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$$(1.5)\quad L(a,b;c)f(z)=z+\sum_{n=2}^{\infty}\frac{(a)_{n-1}(b)_{n-1}}{(n-1)!(c)_{n-1}}a_n z^n, z\in\Delta.$$

We note that L(a,1;c)=L(a;c).Also L(a,1;a)f(z)=f(z) L(2,1;1)f(z)=zf$'(z)$, L(r+1,1;1)f(z)= $D^r f(z)$ where $D^r f(z)$ is theRuscheweyh derivatives of f(z) defined by Ruscheweyh (1975) as

$$(1.6)\qquad D^r f(z)=\frac{z}{(1-z)^{r+1}}*f(z), r>-1.$$

Which is equivalent to

$$D^r f(z)=\frac{z}{r!}\frac{d^r}{dz^r}\{z^{r-1}f(z)\}$$

For $0\le\lambda\le1$,β≥0 and -1≤ α <1, we introduced a subclass T* (α,β,λ) of T consisting of functions f(z) of the form(1.2) and satisfying the condition

$$\operatorname{Re}\left[\frac{z\{L(a,b;c)f(z)\}'}{(1-\lambda)L(a,b;c)f(z)+\lambda z\{L(a,b;c)f(z)\}'}-\alpha\right]>\beta\left|\frac{z\{L(a,b;c)f(z)\}'}{(1-\lambda)L(a,b;c)f(z)+\lambda z\{L(a,b;c)f(z)\}'}-1\right|, z\in\Delta.$$

For b=1, λ=0, T *(α,β,λ) reduces to TS (α,β,) which was defined and studied by G. Murugusundaramoorthy (2004).

In this paper we will obtain a necessary and sufficient conditions for the functions f(z) ∈ T* (α,β,λ). Furthermore extreme points, distortion bounds, Closure properties, radius of starlikeness and convexity for f(z)∈T* (α,β,λ) are also obtained.

COEFFICIENTS INEQUALITY

Theorem2.1: A necessary and sufficient condition for f(z) of the form (1.2) to be in the classT *(α,β,λ) ,-1≤ α <1, β ≥ 0 is that

$$(2.1)\ \sum_{n=2}^{\infty}[n(1+\beta)-\{1+(n-1)\lambda\}(\alpha+\beta)]\frac{(a)_{n-1}(b)_{n-1}}{(n-1)!(c)_{n-1}}|a_n|\le(1-\alpha)$$

Proof: Let f (z) ∈T *(α,β,λ),then it is sufficient to show that

$$\beta\left|\frac{z\{L(a,b;c)f(z)\}'}{(1-\lambda)L(a,b;c)f(z)+\lambda z\{L(a,b;c)f(z)\}'}-1\right|-\operatorname{Re}\left[\frac{z\{L(a,b;c)f(z)\}'}{(1-\lambda)L(a,b;c)f(z)+\lambda z\{L(a,b;c)f(z)\}'}-1\right]\le1-\alpha.$$

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We have

$$\beta\left|\frac{z\{L(a,b;c)f(z)\}'}{(1-\lambda)L(a,b;c)f(z)+\lambda z\{L(a,b;c)f(z)\}'}-1\right|- \mathrm{Re}\left[\frac{z\{L(a,b;c)f(z)\}'}{(1-\lambda)L(a,b;c)f(z)+\lambda z\{L(a,b;c)f(z)\}'}-1\right]$$

$$\leq (1+\beta)\left|\frac{z\{L(a,b;c)f(z)\}'}{(1-\lambda)L(a,b;c)f(z)+\lambda z\{L(a,b;c)f(z)\}'}-1\right|$$

$$\leq (1+\beta)\left|\frac{-\sum_{n=2}^{\infty}\frac{(a)_{n-1}(b)_{n-1}}{(n-1)!(c)_{n-1}}(n-1)(1-\lambda)a_n z^n}{z-\sum_{n=2}^{\infty}\frac{(a)_{n-1}(b)_{n-1}}{(n-1)!(c)_{n-1}}(1+n\lambda-\lambda)a_n z^n}\right|$$

$$\leq (1+\beta)\frac{\sum_{n=2}^{\infty}\frac{(a)_{n-1}(b)_{n-1}}{(n-1)!(c)_{n-1}}(n-1)(1-\lambda)|a_n|}{1-\sum_{n=2}^{\infty}\frac{(a)_{n-1}(b)_{n-1}}{(n-1)!(c)_{n-1}}(1+n\lambda-\lambda)|a_n|}$$

This expression is bounded above by $(1-\alpha)$ if

$$\sum_{n=2}^{\infty}[n(1+\beta)-\{1+(n-1)\lambda\}(\alpha+\beta)]\frac{(a)_{n-1}(b)_{n-1}}{(n-1)!(c)_{n-1}}|a_n| \leq (1-\alpha).$$

Conversely let (2.1) holds. Using the fact that Re $(\omega) > \delta$ if and only if $|\omega-(1+\delta)|<|\omega+(1-\delta)|$,it is enough to show that

$$\left|\frac{z\{L(a,b;c)f(z)\}'}{(1-\lambda)L(a,b;c)f(z)+\lambda z\{L(a,b;c)f(z)\}'}-\left(1+\beta\left|\frac{z\{L(a,b;c)f(z)\}'}{(1-\lambda)L(a,b;c)f(z)+\lambda z\{L(a,b;c)f(z)\}'}-1\right|+\alpha\right)\right|$$

$$<\left|\frac{z\{L(a,b;c)f(z)\}'}{(1-\lambda)L(a,b;c)f(z)+\lambda z\{L(a,b;c)f(z)\}'}+\left(1-\beta\left|\frac{z\{L(a,b;c)f(z)\}'}{(1-\lambda)L(a,b;c)f(z)+\lambda z\{L(a,b;c)f(z)\}'}-1\right|-\alpha\right)\right|$$

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$$Let\ \ E=\left|\frac{z\{L(a,b;c)f(z)\}'}{(1-\lambda)L(a,b;c)f(z)+\lambda z\{L(a,b;c)f(z)\}'}+\left(1-\beta\left|\frac{z\{L(a,b;c)f(z)\}'}{(1-\lambda)L(a,b;c)f(z)+\lambda z\{L(a,b;c)f(z)\}'}-1\right|-\alpha\right)\right|$$

$$=\left|\frac{z-\sum_{n=2}^{\infty}\frac{(a)_{n-1}(b)_{n-1}}{(n-1)!(c)_{n-1}}na_nz^n}{z-\sum_{n=2}^{\infty}\frac{(a)_{n-1}(b)_{n-1}}{(n-1)!(c)_{n-1}}(1+n\lambda-\lambda)a_nz^n}+\left(1-\beta\left|\frac{z-\sum_{n=2}^{\infty}\frac{(a)_{n-1}(b)_{n-1}}{(n-1)!(c)_{n-1}}na_nz^n}{z-\sum_{n=2}^{\infty}\frac{(a)_{n-1}(b)_{n-1}}{(n-1)!(c)_{n-1}}(1+n\lambda-\lambda)a_nz^n}-1\right|-\alpha\right)\right|$$

Putting

$$(1-\lambda)L(a,b;c)f(z)+\lambda z\{L(a,b;c)f(z)\}'=G(z)$$

$$E=\frac{1}{|G(z)|}\left|z-\sum_{n=2}^{\infty}\frac{(a)_{n-1}(b)_{n-1}}{(n-1)!(c)_{n-1}}na_nz^n+(1-\alpha)z-(1-\alpha)\sum_{n=2}^{\infty}\frac{(a)_{n-1}(b)_{n-1}}{(n-1)!(c)_{n-1}}(1+n\lambda-\lambda)a_nz^n-\beta\left|-\sum_{n=2}^{\infty}\frac{(a)_{n-1}(b)_{n-1}}{(n-1)!(c)_{n-1}}(n-1)(1-\lambda)a_nz^n\right|\right|$$

Thus

$$(2.2)E>\frac{|z|}{|G(z)|}\left[(2-\alpha)-\sum_{n=2}^{\infty}\{n+(1-\alpha)(1+n\lambda-\lambda)+\beta(n-1)(1-\lambda)\}\frac{(a)_{n-1}(b)_{n-1}}{(n-1)!(c)_{n-1}}a_n\right]$$

$$\text{Again let }F=\left|\frac{z\{L(a,b;c)f(z)\}'}{(1-\lambda)L(a,b;c)f(z)+\lambda z\{L(a,b;c)f(z)\}'}-\left(1+\beta\left|\frac{z\{L(a,b;c)f(z)\}'}{(1-\lambda)L(a,b;c)f(z)+\lambda z\{L(a,b;c)f(z)\}'}-1\right|+\alpha\right)\right|$$

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$$= \frac{1}{|G(z)|}\left| \begin{array}{l} z - \sum_{n=2}^{\infty}\frac{(a)_{n-1}(b)_{n-1}}{(n-1)!(c)_{n-1}} n a_n z^n - (1+\alpha)z \\ + (1+\alpha)\sum_{n=2}^{\alpha}\frac{(a)_{n-1}(b)_{n-1}}{(n-1)!(c)_{n-1}}(1+n\lambda-\lambda)a_n z^n - \beta\left| -\sum_{n=2}^{\infty}\frac{(a)_{n-1}(b)_{n-1}}{(n-1)!(c)_{n-1}}(n-1)(1-\lambda)a_n z^n \right| \end{array} \right|$$

$$= \frac{1}{|G(z)|}\left| \begin{array}{l} -\alpha z - \sum_{n=2}^{\infty}\{n-(1+n\lambda-\lambda)(1+\alpha)\}\frac{(a)_{n-1}(b)_{n-1}}{(n-1)!(c)_{n-1}} a_n z^n \\ -\sum_{n=2}^{\infty}\beta(n-1)(1-\lambda)\frac{(a)_{n-1}(b)_{n-1}}{(n-1)!(c)_{n-1}} a_n z^n \end{array} \right|$$

$$\text{Thus}\,(2.3) F < \frac{|z|}{|G(z)|}\left[\alpha + \sum_{n=2}^{\infty}\{n-(1+n\lambda-\lambda)(1+\alpha)+\beta(n-1)(1-\lambda)\}\frac{(a)_{n-1}(b)_{n-1}}{(n-1)!(c)_{n-1}} a_n \right]$$

Now, from (2.2), (2.3), it follows that (2.4)

$$(2.4) E - F > \frac{2|z|}{|G(z)|}\left[(1-\alpha) - \sum_{n=2}^{\infty}\{n(1+\beta)-\{1+(n-1)\lambda\}(\alpha+\beta)\}\frac{(a)_{n-1}(b)_{n-1}}{(n-1)!(c)_{n-1}}|a_n|\right]$$

Thus (2.1) proves the theorem.

The result is sharp. The extremal function being

$$(2.5)\, f(z) = \frac{\{n(1+\beta)-\{1+(n-1)\lambda\}(\alpha+\beta)\}\frac{(a)_{n-1}(b)_{n-1}}{(n-1)!(c)_{n-1}} z - (1-\alpha)z^n}{\{n(1+\beta)-\{1+(n-1)\lambda\}(\alpha+\beta)\}\frac{(a)_{n-1}(b)_{n-1}}{(n-1)!(c)_{n-1}}}, n \geq 2.$$

DISTORTION THEOREMS

Theorem 3.1: If f (z) ∈T *(α,β,λ), then for |z|= r < 1

$$(3.1)\left(r - \frac{(1-\alpha)c}{\{2(1+\beta)-(1+\lambda)(\alpha+\beta)\}ab} r^2\right) \leq |f(z)| \leq \left(r + \frac{(1-\alpha)c}{\{2(1+\beta)-(1+\lambda)(\alpha+\beta)\}ab} r^2\right)$$

and

$$(3.2)\left(r - \frac{(1-\alpha)}{\{2(1+\beta)-(1+\lambda)(\alpha+\beta)\}} r^2\right) \leq |L(a,b;c) f(z)| \leq \left(r + \frac{(1-\alpha)}{\{2(1+\beta)-(1+\lambda)(\alpha+\beta)\}} r^2\right)$$

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Proof: In view of inequality (2.1), it follows that

$$\sum_{n=2}^{\infty}[n(1+\beta)-\{1+(n-1)\lambda\}(\alpha+\beta)]\frac{(a)_{n-1}(b)_{n-1}}{(n-1)!(c)_{n-1}}a_n \le (1-\alpha).$$

By the fact that $\frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}}$ is non-decreasing for n $\ge$ 2.Then

$$\{2(1+\beta)-(1+\lambda)(\alpha+\beta)\}\frac{ab}{c}\sum_{n=2}^{\infty}a_n \le \sum_{n=2}^{\infty}[n(1+\beta)-\{1+(n-1)\lambda\}(\alpha+\beta)]\frac{(a)_{n-1}(b)_{n-1}}{(n-1)!(c)_{n-1}}a_n$$

$$\le (1-\alpha).$$

or, $\sum_{n=2}^{\infty}a_n \le \frac{(1-\alpha)c}{\{2(1+\beta)-(1+\lambda)(\alpha+\beta)\}ab}$

Therefore

(3.3) $$|f(z)| \ge \left(r - \frac{(1-\alpha)c}{\{2(1+\beta)-(1+\lambda)(\alpha+\beta)\}ab}r^2\right)$$

and

(3.4) $$|f(z)| \le \left(r + \frac{(1-\alpha)c}{\{2(1+\beta)-(1+\lambda)(\alpha+\beta)\}ab}r^2\right)$$

From (3.3) and (3.4) inequality (3.1) follows.

Further, for f (z) ∈T *(α,β,λ),inequality (2.1) gives

$$\{2(1+\beta)-(1+\lambda)(\alpha+\beta)\}\sum_{n=2}^{\infty}\frac{(a)_{n-1}(b)_{n-1}}{(n-1)!(c)_{n-1}}a_n \le (1-\alpha).$$

Or, $\sum_{n=2}^{\infty}\frac{(a)_{n-1}(b)_{n-1}}{(n-1)!(c)_{n-1}}a_n \le \frac{(1-\alpha)}{\{2(1+\beta)-(1+\lambda)(\alpha+\beta)\}}$

Thus,

(3.5) $$|L(a,b;c)f(z)| \ge \left(r - \frac{(1-\alpha)}{\{2(1+\beta)-(1+\lambda)(\alpha+\beta)\}}r^2\right)$$

and

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$$(3.6)\qquad |L(a,b;c)f(z)| \le \left(r + \frac{(1-\alpha)}{\{2(1+\beta)-(1+\lambda)(\alpha+\beta)\}} r^2 \right)$$

On using (3.5) and (3.6) inequality (3.2) follows.

Remark3.2: The bounds in (3.1) & (3.2) are sharp, since the inequalities are attained for the function.

$$(3.7)\quad f(z) = \frac{\{2(1+\beta)-(1+\lambda)(\alpha+\beta)\}abz-(1-\alpha)cz^2}{\{2(1+\beta)-(1+\lambda)(\alpha+\beta)\}ab},$$

EXTREME POINTS

Theorem 4.1: Let

$$(4.1)\quad f_1(z) = z \text{ and } \qquad f_n(z) = a_1 z - \frac{(1-\alpha)}{\{n(1+\beta)-\{1+(n-1)\lambda\}(\alpha+\beta)\}\frac{(a)_{n-1}(b)_{n-1}}{(n-1)!(c)_{n-1}}} z^n$$

for n≥2, then f (z) ∈T *(α,β,λ), if and only if it can be expressed in the form

$$(4.2)\ f(z) = \sum_{n=1}^{\infty} d_n f_n(z), \text{ where } d_n \ge 0 \text{ and } \sum_{n=1}^{\infty} d_n = 1.$$

In particular the extreme points of T *(α,β,λ) are the functions given by (4.1).

Proof: Let f(z) be expressed in the form (4.1),then

$$f(z) = \sum_{n=1}^{\infty} d_n f_n(z) = z - \sum_{n=2}^{\infty} \frac{(1-\alpha)d_n}{\{n(1+\beta)-\{1+(n-1)\lambda\}(\alpha+\beta)\}\frac{(a)_{n-1}(b)_{n-1}}{(n-1)!(c)_{n-1}}} z^n$$

$$= z - \sum_{n=2}^{\infty} d_n t_n z^n$$

Where $t_n = \dfrac{(1-\alpha)}{\{n(1+\beta)-\{1+(n-1)\lambda\}(\alpha+\beta)\}\frac{(a)_{n-1}(b)_{n-1}}{(n-1)!(c)_{n-1}}}$

Now, since

$$\sum_{n=2}^{\infty} \{n(1+\beta)-\{1+(n-1)\lambda\}(\alpha+\beta)\}\frac{(a)_{n-1}(b)_{n-1}}{(n-1)!(c)_{n-1}} d_n t_n = \sum_{n=2}^{\infty} (1-\alpha)d_n$$

$$= (1-\alpha)(1-d_1) \le (1-\alpha).$$

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Therefore, f (z) ∈T *(α,β,λ).

Conversely, let f (z) ∈T *(α,β,λ),then(2.1) yields

$$a_n \leq \frac{(1-\alpha)}{\{n(1+\beta)-\{1+(n-1)\lambda\}(\alpha+\beta)\}\frac{(a)_{n-1}(b)_{n-1}}{(n-1)!(c)_{n-1}}} z^n \text{ for } n \geq 2.$$

Setting $d_n = \frac{[n(1+\beta)-\{1+(n-1)\lambda\}(\alpha+\beta)](a)_{n-1}(b)_{n-1}}{(n-1)!(c)_{n-1}(1-\alpha)} a_n$ for $n \geq 2$

and $d_1 = 1-\sum_{n=2}^{\infty} d_n$.

$$\text{Then } f(z) = z - \sum_{n=2}^{\infty} \frac{(1-\alpha)}{\{n(1+\beta)-\{1+(n-1)\lambda\}(\alpha+\beta)\}\frac{(a)_{n-1}(b)_{n-1}}{(n-1)!(c)_{n-1}}} d_n z^n$$

$$= z - \sum_{n=2}^{\infty} d_n \{z - f_n(z)\}$$

$$= z(1-\sum_{n=2}^{\infty} d_n) + \sum_{n=2}^{\infty} d_n f_n(z) = \sum_{n=1}^{\infty} d_n f_n(z).$$

This completes the proof.

RADIUS OF STARLIKENESS

THEOREM 5.1: Let f (z) ∈T *(α,β,λ),then f(z) is starlike in $|z| < r(\alpha,\beta,\lambda)$,where

$$(5.1)\, r = \inf\left[\frac{[n(1+\beta)-\{1+(n-1)\lambda\}(\alpha+\beta)](a)_{n-1}(b)_{n-1}}{(n-1)!(c)_{n-1}(1-\alpha)n}\right]^{\frac{1}{n-1}}, n \geq 2,$$

Proof: It suffices to show that

$$\left|\frac{zf'(z)}{f(z)} - 1\right| < 1$$

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i.e. $\left|\frac{zf'(z)}{f(z)}-1\right| \le \frac{\sum_{n=2}^{\infty}(n-1)a_n|z|^{n-1}}{1-\sum_{n=2}^{\infty}a_n|z|^{n-1}} < 1$

(5.2) or $\sum_{n=2}^{\infty} na_n|z|^{n-1} < 1$

It is easily to see that (5.1) holds if

$$|z|^{n-1} < \left[\frac{[n(1+\beta)-\{1+(n-1)\lambda\}(\alpha+\beta)](a)_{n-1}(b)_{n-1}}{(n-1)!(c)_{n-1}(1-\alpha)n}\right].$$

This completes the proof.

RADIUS OF CONVEXITY

THEOREM 6.1: Let f (z) ∈T *(α,β,λ),then f(z) is convex in $|z| < r(\alpha,\beta,\lambda)$,where

$$(6.1)\quad r = \inf\left[\frac{[n(1+\beta)-\{1+(n-1)\lambda\}(\alpha+\beta)](a)_{n-1}(b)_{n-1}}{(n-1)!(c)_{n-1}(1-\alpha)n^2}\right]^{\frac{1}{n-1}}, n \ge 2.$$

Proof: Upon noting the fact that f(z) is convex if and only if $zf'(z)$is starlike, the Theorem(6.1) follows.

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